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# Kinematics of a Rolling Tire and Its Application to Tire Performance

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## I. INTRODUCTION

In analysis of the mechanics of a rolling tire, the assumption is usually made that the wheel contacts the ground at a single point. In practice, the tire contacts the ground over some finite length. The motion and strains which take place along this length are paramount in determining the wear, traction, and rolling resistance which occur during rolling. The purpose of this paper is to derive in detail the mechanics of what happens over the contact length.

In the analytic work, a two-dimensional wheel will be considered. The approach will be purely kinematical or geometrical. Some of the dynamic consequences will be discussed at the conclusion of the analysis.

Two conditions will be considered, that of a free-rolling and that of a driven wheel. For each of these conditions, studies will be made of two types of wheels: (a) a wheel of high torsional rigidity and low tread compressibility, and (b) a wheel of high tread compressibility and low torsional rigidity.

## II. DISCUSSION

### Section 1

A wheel rolling on the road may be considered in two ways: as a wheel rolling along the ground which is considered stationary; or the center of the wheel may be taken as fixed, so that the ground moves relative to it. Either case, of course, will lead to the correct results. In this analysis, the center of the wheel will be held fixed while the ground will be considered to move with a velocity $\nu$. The premise of the first part of the discussion is that the road will travel a distance equal to the contact length; ($2l$), each time the tire rotates through an angle equal to the central angle, $2\theta_0$, of the contact length. This point has been checked experimentally on a free-rolling wheel and found to be a fact. For the present we will also assume that radial lines cannot bend (infinite torsional rigidity), though they can compress. We will also assume the tread can compress or extend as the motion requires. Now consider a wheel as shown in Figure 1. When the tire

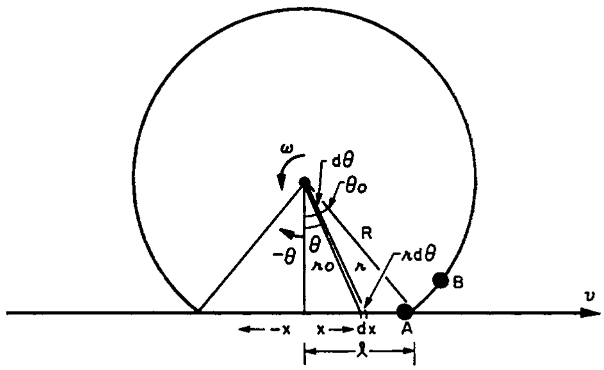


Fig. 1. Rolling wheel.

rotates through an angle $d\theta$ ($\theta = \theta_0$ minus the angle traversed), the tread at the point $x$ moves a distance $dx$ and

$$r_0 \, d\theta = dx \cos \theta \tag{1}$$

$$dx = (r_0/\cos^2 \theta) d\theta$$

where $r$ is the distance from the center of the wheel to a point along the contact and $r_0$ is the standing height of the tire.

$$dx/dt = (r_0/\cos^2 \theta)(d\theta/dt) \tag{2}$$

but $\omega = (d\theta/dt)$ is the angular velocity of the wheel; hence:

$$dx/dt = (r_0/\cos^2 \theta)\omega \tag{3}$$

Note that the tread, upon entering, is traveling at a speed faster than the peripheral speed out of contact. That is, the tread at point A is traveling faster than the tread at point B (see Fig. 1). This introduces a discontinuity. In practice, the change in velocity probably takes place over a finite distance outside but adjacent to the contact length, as well as within the contact length. Now if the road is traveling at a velocity $\nu$, the relative (to the road) velocity $V_r$ of the point $x$ on the periphery of the wheel is:

$$V_r = (dx/dt) - \nu = (r_0\omega/\cos^2\theta) - \nu \qquad (4)$$

(When $V_r$ is positive, the tread is traveling faster than the road, and when it is negative the tread is traveling slower.)

We stated earlier that when the tire traverses an angle of $2\theta_0$, the road travels a distance $2l$. For constant angular velocity, the time regained to rotate $2\theta_0$ is given by $t = 2\theta_0/\omega$; hence the velocity of the road must be

$$\nu = 2l/t = l\omega/\theta_0 \qquad (5)$$

or

$$\omega = \nu\theta_0/l$$

Substituting eq. (5) in eq. (4), we find that the relative velocity at $x$ is

$$V_r = (r_0\nu\theta_0/l\cos^2\theta) - \nu = \nu[(r_0\theta_0/l\cos^2\theta) - 1] \qquad (6)$$

But, since $r_0/l = 1/\tan\theta_0$, this yields

$$V_r = [(\theta_0/\tan\theta_0)(1/\cos^2\theta) - 1]\nu \qquad (7)$$

Entering contact ($\theta = -\theta_0$), the tread is traveling faster than the road. As it proceeds through contact, it slows up and, at some point its speed is equal to road speed. Beyond that point the tread continues to slow up until it reaches the center of contact. It now begins to speed up, once again reaching road speed; at contact exit ($\theta = \theta_0$) the tread is again traveling faster than the road and at the same speed as at entrance. This concept has been discussed qualitatively.[1]

In Figure 2 we have plotted the relative velocity at different points along the contact length for a

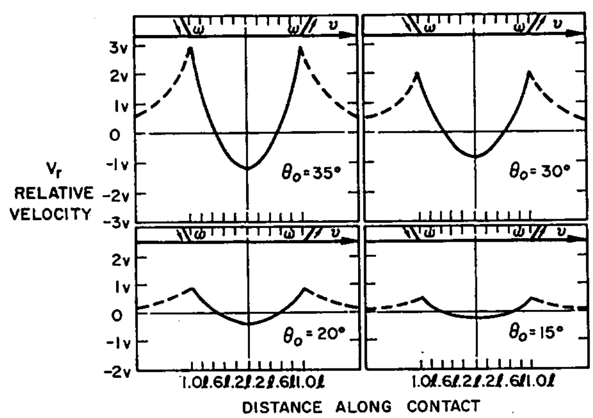


Fig. 2. Relative velocity between tread and road.

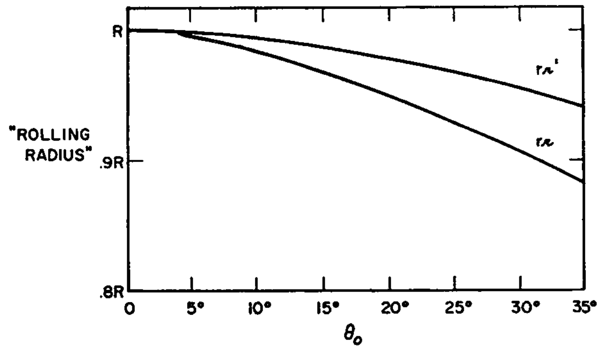


Fig. 3. Rolling radius $r_r'$ ($-\nu$/w) vs. $\theta_0$ and rolling radius at zero relative velocity $r_r$ vs. $\theta_0$.

series of deflections. By deflection of the tire we mean a decrease in $r_0$ or an increase in $\theta_0$. This is tantamount to increasing the vertical load on the tire. The relative velocity is, in a sense, a measure of slip. We find that the ends of contact are points of high slip. Also, as would be expected, the amount of slipping increases with the severity of the deflection. This slip, though not causing the wheel to spin, can result in wear, noise, and friction losses. Also, the variation of slip imposes compression along the tread. The details of this compression will be studied later in the discussion.

In order to understand how a wheel rolls, we should examine the points of zero velocity. In the undeflected wheel, the zero velocity point is the point about which rolling is taking place. The extension of this definition to the deflected case is that the wheel rotates about the two zero velocity points and the rolling radii $r_r$ could be considered to be the radii from the center of the wheel to the zero points. These radii are equal. From eq. 7, the angle of either of these radii is given by

$$\cos^2\theta = \theta_0/\tan\theta_0 \qquad (8)$$

Since

$$r = R\cos\theta_0/\cos\theta \qquad (9)$$

(where $R$ is the free tire radius), the rolling radii are given by

$$r_r = R\cos\theta_0/(\theta_0/\tan\theta_0)^{1/2} = R(\sin 2\theta_0/2\theta_0)^{1/2} \qquad (10)$$

A plot of $r_r$ is shown in Figure 3.

In an undeflected wheel, the rolling radius can also be defined as the ratio of the road speed to the angular velocity of the wheel. When this definition is applied to the deflected wheel, we find a "rolling radius" that is different from the zero velocity

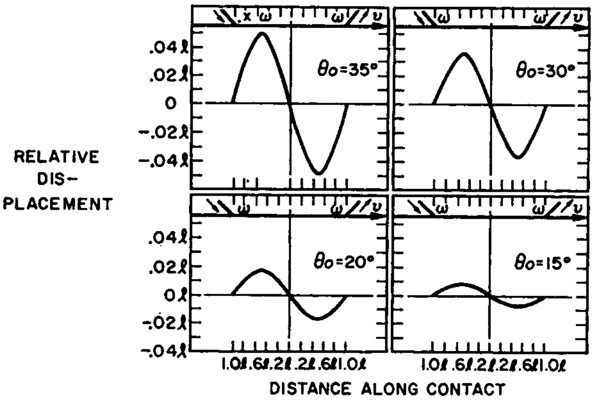

Fig. 4. Relative displacement between tread and road.

"rolling radius" described above. The new "rolling radius" $r_r'' = \nu/\omega$, from eq. (5), is

$$r_r = l/\theta_0 = R \sin \theta_0/\theta_0 \tag{11}$$

Thus, this analysis of the rolling of a deflected wheel indicates there are four radii one must consider: (1) $R$, the free radius of the wheel; (2) $r_0$, the standing height of the wheel; (3) $r_r$, the zero velocity radius; and (4) $r_r'$, the ratio of the road velocity to the angular velocity. The latter two are plotted in Figure 3. In an undeflected wheel, of course, the four radii are equal.

Now let us consider the slip in detail. A point on the tread contacts a point on the road. Since the tread is traveling faster than the road, the point on the tread passes the point on the road. At the center of contact the tread has slowed up so that the two points coincide. Then the road point passes the tread. The tread speeds up and at the end of contact they are together again. The relative motion between these two points represents the total slip going through contact. Then the relative displacement $dx_r$ between these two points going through contact (where $x_r$ is distance traversed by a contact point relative to which it touches upon entering the contact length) is

$$dx_r = V_r dt = \nu[(\theta_0/\tan \theta_0)(1/\cos^2\theta) - 1]dt \tag{12}$$

Upon integration, we obtain for $x_r$

$$x_r = l[(\tan \theta/\tan \theta_0) - (\theta/\theta_0)] \tag{13}$$

When $x_r$ is positive, the tread is traveling ahead of the road; when it is negative, the tread is behind the corresponding point on the road. The variation of $x_r$ along the contact is shown in Figure 4. From this relation we see that the tire tread executes a type of harmonic motion relative to the road. It is obvious that the maximum relative displacement occurs at the zero velocity point, as can be seen by differentiating eq. (13) and then setting it equal to zero.

Since each point in contact is traveling at a velocity different from its neighboring points, it is obvious that the tread must compress and expand along the circumference in order to accommodate this motion. This compression can easily be obtained from the geometry. The original length of any element is $R/d\theta$. In contact, its length is $dx$. Thus, the per cent compression or extension given by

$$\begin{aligned} E &= 100[(Rd\theta - dx)/Rd\theta] \\ &= 100\{[Rd\theta - (r_0 d\theta/\cos^2 \theta)]/Rd\theta\} \\ &= 100\ [(\cos^2 \theta - \cos \theta_0)/\cos^2 \theta] \end{aligned} \tag{14}$$

A plot of compression is shown in Figure 5. This has been plotted in such a way that negative values signify compression and positive values, extension [note that in equation (14) that negative $E$ signifies extension]. We see that the rubber (in a tire tread) enters contact in an extended state, proceeds through relative compression until it reaches zero, and then goes into absolute compression. Maximum compression occurs at the center of contact. From that point, it starts to expand and finally leaves contact in the extended state. The rubber has zero compression or extension at a position defined by

$$\cos^2 \theta = \cos \theta_0 \tag{15}$$

At this position the tread in contact is traveling at the same speed as the tread out of contact; this is shown as follows. From eq. (3), the speed in contact is

$$dx/dt = (r_0/\cos^2 \theta)\omega$$

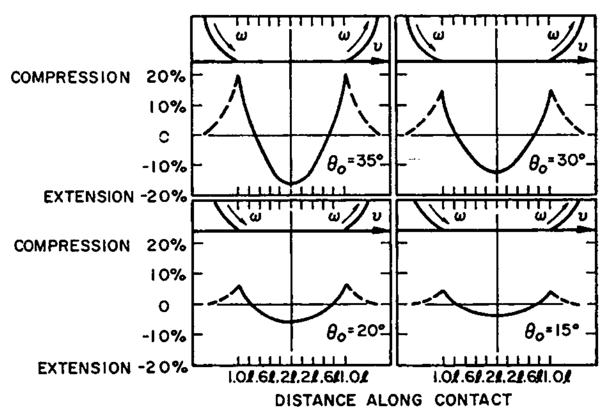

Fig. 5. Compression and extension.

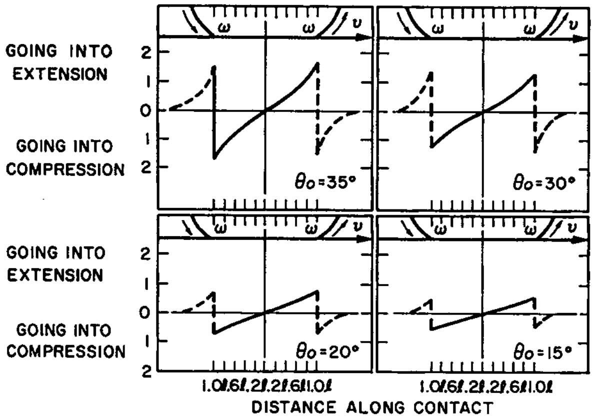


Fig. 6. Rate of change of compression.

The speed out of contact is

$$R\omega = (r_0/\cos\theta_0)\omega$$

Equating these two gives

$$(r_0/\cos^2\theta)\omega = r_0\omega/\cos\theta_0 \qquad \text{or} \qquad \cos^2\theta = \cos\theta_0$$

Another important quantity is the rate of change of compression along the contact length. This is a measure of how fast the tread material must adjust in order to accommodate the motion. From eq. (14),

$$dE/d\theta = 200(\cos\theta_0 \sin\theta/\cos^3\theta)$$

The changes in compression (Fig. 6) to occur rapidly at the entrance and exit of contact.

## Section 2

In the previous discussion it was assumed that the compressional modulus of the tread was much lower than the torsional modulus of the structure. We will now assume the converse, that the wheel adjusts to the motion primarily by a bending of the radial lines in the contact region. In order for this to happen, the tread in contact must not move relative to the road. Hence, the tread is traveling at the road velocity given by eq. (5). The distance traveled, therefore, is obtained by integration of eq. (5):

$$x = l\theta/\theta_0 \tag{16}$$

The bending that this imposes is described as follows. As a point enters contact, the radius associated with this point is straight. Going through contact, this radial line bends toward the center of contact. The bending reaches a maximum at the "zero velocity" point and then reduces until there is zero bending at the center of contact. As the point goes through the second half of contact, the radial line bends in the same manner reaching zero bending at exit. An example of this motion is shown in Figure 7.

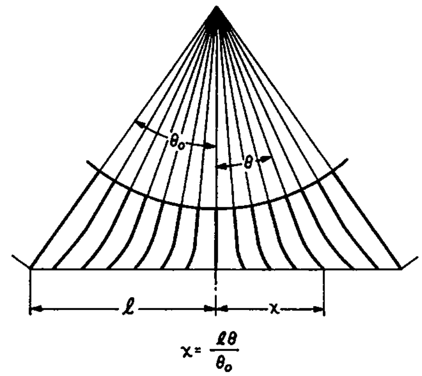


Fig. 7. Bending and no slip.

The compression which the tread must undergo to accommodate this motion is obviously

$$E = 100[(R\theta_0 - l)/R\theta_0]$$
$$= 100[(\theta_0 - \sin\theta_0)/\theta_0] \tag{17}$$

This compression is uniform throughout contact. From geometry it follows that the average compression of the tread as described in the earlier situation and as defined by eq. (14) must equal the compression defined by eq. (17). This is shown as follows.

The average compression in the first case is (see eq. 14)

$$\bar{E} = (100/\theta_0)\int_0^{\theta_0}[(\cos^2\theta - \cos\theta_0)/\cos^2\theta]d\theta$$

$$= 100[(\theta_0 - \sin\theta_0)/\theta_0] \tag{18}$$

which is identical to that given by eq. (17) for the second case.

## Section 3

The two cases described above are obviously extreme cases. In a tire, probably neither case arises exclusive of the other. In Section 1, the situation is such that the tread and associated structure tend to take up all the deformation in compression; this results in slipping throughout most of contact. In Section 2, the deformation is taken up primarily by torsion or bending and a slight amount of compression. [Comparison of eqs. (17) and (14) shows that the compression and extension demanded by the situation in Section 1 is much greater than that in Section 2.] In practice, both phenomena probably take place. Thus, there would be a zone around the zero velocity points where there would be no slipping but some bending, while at the ends and perhaps in the center of contact, we would expect slipping as described by the equations in Section 1.

The curves and expressions derived thus far indicate that the contact region affects at least the material of the wheel immediately outside it. Continuity demands both an extension of the tread and velocities higher than the peripheral velocity at regions immediately adjacent to the contact length. In the case where bending accommodates the tire motion, we would expect in the regions adjacent to contact (1) velocities slower than the peripheral velocity and (2) a tread compression. In the various figures, the regions adjacent to the contact length have been sketched in with dashed lines. These dashed lines represent a possible representation which is consistent with the analysis of the contact region.

## Section 4

The above discussion is applicable only to a free-rolling wheel at constant velocity having no gross slip. However, the analysis can be carried further so as to include the kinematics of a wheel which has gross slip and is running at constant velocity. This can be considered a "driven wheel"

As before, let us consider first a wheel which permits no bending, but can compress readily along the tread. When the wheel rotates an angle $2\theta_0$, the distance traveled by the road will be determined by the amount of gross slip. (If gross slip is zero, the distance traveled is, of course, $2l$.) Let us call this distance $Z$. Then if $t_1$ is the time required for the wheel to rotate $2\theta_0$, the angular velocity of the wheel $\omega$ and the linear velocity of the road $\nu$ are:

$$\omega = 2\theta_0/t_1$$

and

$$\nu = Z/t_1$$

Hence,

$$\omega = 2\theta_0\nu/Z \tag{19}$$

As before, the distance traveled by a point on the tread is

$$dx = (r_0/\cos^2\theta)d\theta \tag{20}$$

and the tread velocity is

$$dx/dt = (r_0/\cos^2\theta)(d\theta/dt) = (r_0/\cos^2\theta)\omega \tag{21}$$

The relative velocity $V_r$ between the road and the tread is given by

$$V_r = (dx/dt) - \nu = (r_0\omega/\cos^2\theta) - \nu \tag{22}$$

Using eq. (19), we obtain

$$V_r = \nu[(2l\theta_0/Z\tan\theta_0)(1/\cos^2\theta) - 1] \tag{23}$$

For this discussion we shall rewrite $V_r$ in somewhat different terms than we did in the case of no gross slip. When the tire slips, the road travels a distance less than $2l$. This distance is of course $Z$. Let us relate $Z$ and $2l$ as follows:

$$Z = n2l \quad \text{where} \quad 0 < n \le 1 \tag{24}$$

Then we may define the per cent slip $P$ as:

$$100[(2l - 2nl/2l)] = 100(1 - n) = P \tag{25}$$

Substituting $n$ in eq. (23), we have

$$V_r = \nu[(\theta/n\tan\theta_0)(1/\cos^2\theta) - 1] \tag{26}$$

The relative velocity will be zero when

$$\cos^2\theta = \theta_0/n\tan\theta_0 \tag{27}$$

$\theta$ decreases as $n$ decreases, i.e., as the per cent slip increases, but the smallest $\theta$ can be is zero. This obviously sets the minimum for $n$:

$$1 = \theta_0/n_{\min}\tan\theta_0 \quad \text{or} \quad n_{\min} = \theta_0/\tan\theta_0 \tag{28}$$

This means that when the per cent slip is greater than shown by eq. (28), (i.e., $n$ is less than $n_{\min}$), the tire is no longer actually "rolling." Every point in contact is moving faster than the road and the wheel is just spinning over the road. The analysis does not hold for this case. We will only consider cases for which

$$\theta_0/\tan\theta_0 \le n \le 1$$

From eq. (28), we have:

$$P_{\max} = 100[1 - (\theta_0/\tan\theta_0)] \tag{29}$$

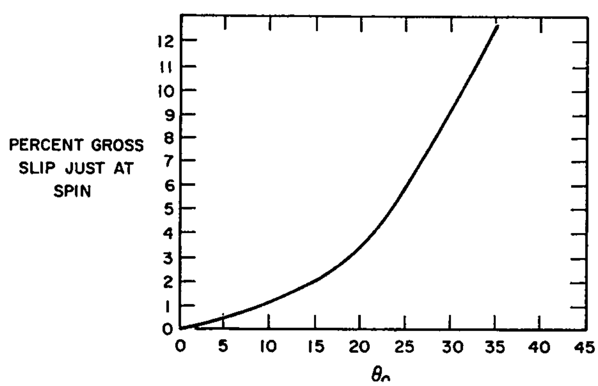


Fig. 8. Per cent gross slip just at spin vs. $\theta_0$.

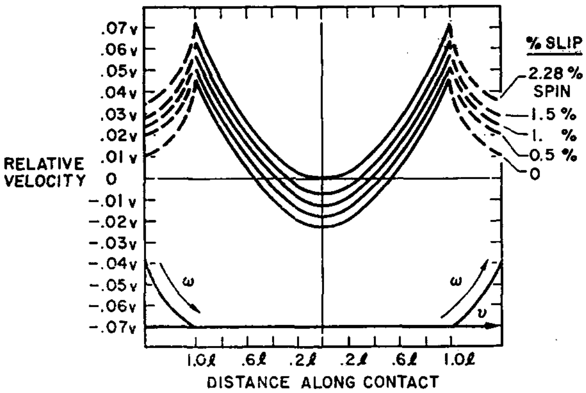


Fig. 9. Relative velocity under gross slip conditions.

A plot of this equation is shown in Figure 8. From this figure we see that the greater the deflection the more slip can take place before the wheel is forced into spin.

In Figure 9, we have plotted the relative velocity for various values of $n$, at $\theta_0 = 15°$; we see that the effect of slip is to simply increase the relative velocity of each point in contact. A more significant relationship is obtained from the relative displacement. As explained earlier, the relative displacement is the distance between a point on the tread, as it traverses the contact length, and the point on the road it first contacted when it entered contact. From eq. (26),

$$V_r = dx_r/dt = \nu[(\theta_0/n \tan \theta_0)(1/\cos^2 \theta) - 1] \quad (26)$$

This becomes, upon integration,

$$x_r = l[(\tan \theta/\tan \theta_0 + 1) + (\theta/\theta_0 + 1)n] \quad (30)$$

Again, when $x_r$ is positive, the tread is ahead of the corresponding point on the road.

Plots for $x_r$ for various values of $n$ at $\theta_0 = 15°$ are shown in Figure 10.

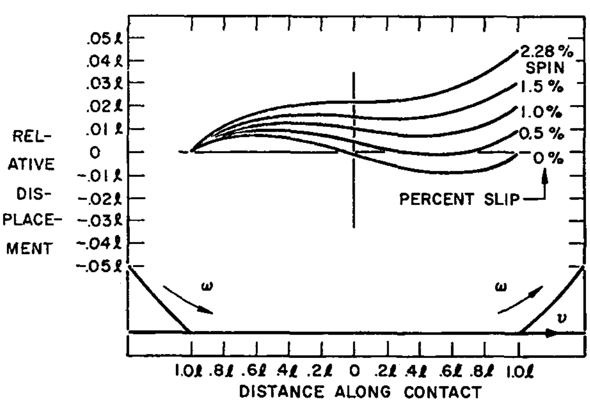


Fig. 10. Relative displacement under gross slip conditions.

Now since these are curves of relative displacement, the amount of slip between any two points in contact is simply the difference in the relative displacement. As the per cent gross slip is increased, the slip tends to localize itself at the beginning and end of contact. Thus the slip rate increases with amount of gross slip. This occurs because the slip becomes more localized as the gross slip increases, hence the time in which the slip occurs becomes shorter with increased gross slipping.

## Section 5

If the wheel can exhibit torsional flexibility so that no slip takes place under conditions described above, the following must occur. The position of a point on the tread must stay with the point on the

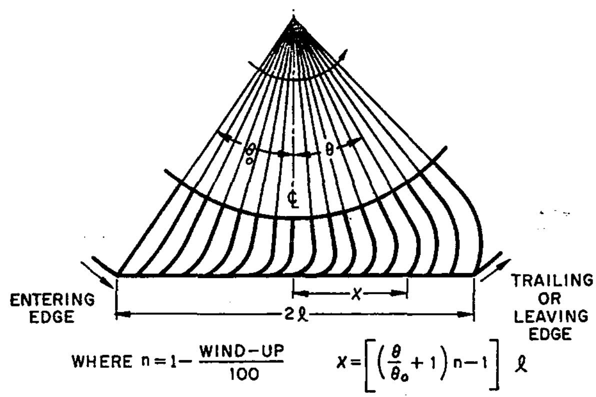


Fig. 11. Bending and no slip; 10% "wind-up."

road. Thus, the position of a point on the tread must equal

$$\bar{x} = x - x_r \quad (31)$$

where $x$ is simply the position of the point in the unbent case and $x_r$ is the relative displacement between tread and road which would occur if slipping took place. Using $x = l(\tan \theta/\tan \theta_0)$ and $x_r$ from eq. (30), we obtain:

$$\bar{x} = l[(\theta/\theta_0 + 1)n - 1] \quad (32)$$

The bending indicated by eq. (32) is shown in Figure 11. In order to show the effect, Figure 11 represents a very extreme case. Since there is no slip, this bending must represent an increase in the angular velocity relative to the road velocity and a subsequent wind-up in the contact length. The term "wind-up" means that the *angular* velocity around the center of the wheel decreases as we move toward the circumference of the wheel in the contact region. That this must be so is shown by the fact (see Fig. 11) that there is an increase of the central angle of about 10%. Hence, in the same time the tire traverses contact length, the tire rotates through an angle 10% greater than the central angle. The trailing edge must relieve the wind-up as it comes out of contact.

### III. EFFECT ON TIRE PERFORMANCE AND DESIGN

We will now attempt to apply the previous analysis to some performance and design characteristics of tires. The foregoing analysis is limited, in as much as it is strictly two-dimensional. In a tire we have four or five rows of treads. Each row represents a different geometry and hence each has to be treated separately. That is to say, as we examine the tire laterally, some sections might be slipping and others might be holding and bending. This undoubtedly results in lateral forces and slipping. We will not discuss these effects here.

It would appear that there are two important properties of tires which contribute to their characteristics. One is circumferential stiffness or compressibility and the other is the torsional rigidity of the tire. It should be possible to make measurements of these quantities on tires and thus be better able to predict the performance of the tires. Furthermore, an understanding of the factors which contribute to the torsional rigidity and circumferential stiffness should lead to designs which can have desired performance characteristics.

Unfortunately, many factors which make for low torsional rigidity also result in low circumferential stiffness. However, by judicious choice of tread design, polymer, carcass, etc., I believe the parameters can be varied over a large range. It should be stressed that the various design features are interdependent, as far as determining the two moduli are concerned. Thus, when one design feature is changed, it is often necessary to change others so as to keep the overall tire characteristics constant.

For most tire performance factors, it would seem that a tire with low torsional rigidity and high circumferential rigidity would be desirable. As the analysis shows, this type of tire undergoes very little scrubbing in contact. This should make for low rolling resistance (due to scrubbing), low tread wear, and good skid and traction. Its disadvantages are poor stability, possible tearing of tread buttons, more rolling resistances due to hysteresis of material, and possibly greater wear on the driving wheel than on the rolling wheels.

The effect of load, inflation, and deflection tire performance can be obtained from the analysis. However, these operating variables are set by automotive manufacturers and very little variation is permitted in actual operations. Tire manufacturers do have the widest latitude in using new materials and designs. As the analysis shows, the correct choice can lead to tires having almost any desirable characteristics.

The author is indebted to H. H. Vickers and H. C. North for the valuable advice and time spent in discussions concerning this work.

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## Synopsis

A geometrical analysis is given of the tread motions occurring in the contact length of a rolling tire. Both the case of a free rolling wheel and that of a wheel under torque are discussed. We have derived expressions and curves for the relative slip and compressions which occur at the contact length for the two conditions. It is found that two important structural features control the nature of tread motions. These are the torsional rigidity of the sidewall and the compressibility of the tread. In addition, the implication of the analysis in tire design and performance are discussed.

## Résumé

On donne une analyse géométrique des mouvements qui se présentent par suite de contacts longitudinaux dans un pneu en roulement. Les deux cas ont été envisagés, savoir celui d'une roue libre et d'une roue soumise à une certaine tension. Des expressions et courbes ont été dérivées pour le glissement relatif et les compressions qui se présentent par contact longitudinal dans ces deux conditions. Deux phénomènes structuraux importans controlent la nature de ces mouvements. Ce sont la rigidité torsionnelle latérale et al compressibilité. En outre on discute de l'influence de cette analyse sur le dessin et la performance ultérieure du pneu.

## Zusammenfassung

Eine geometrische Analyse der in der Kontaktlänge eines rollenden Reifens ablaufenden Bewegungen der Lauffläche wird durchgeführt. Es wird sowohl der Fall eines frei rollenden, als auch der eines angetriebenen Rades diskutiert. Ausdrücke und Kurven für das relative Gleiten und die Kompressionen in der Kontaktlänge wurden für beide Bedingungen abgeleitet. Es wurde gefunden, dass für die in der Lauffläche auftretenden Bewegungen zwei wichtige strukturelle Grössen ausschlaggebend sind. Es sind dies die Torsionsfestigkeit der Seitenwand und die Kompressibilität der Lauffläche. Schliesslich werden die sich aus der Analyse für den Entzurf und die Leistungsfähigkeit von Reifen ergebenden Folgerungen diskutiert.

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